Pricing Redundant Assets In A Complete Market

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Arbitrage is defined as the ability to earn a positive return at no risk on a zero investment. By creating such a portfolio an investor would receive at no cost the possibility of receiving money in the future. In competitive market economies arbitrages are not permitted to exist. An arbitrage portfolio has the following characteristics...

1 Has zero cost to set up
2 Has non-negative values in the future
3 May be of positive value in the future

We will define a basis asset to be a security with a known payoff vector at time t and a known price at time zero. The matrix of basis asset payoff vectors may or may not be linearly independent. We will define a complete market to be a market in which there are m basis assets and m states of the world such that the matrix of basis asset payoff vectors spans $\mathbb{R}^3$ (i.e. the matrix of payoff vectors is linearly independent). We will define a focus asset to be a redundant security in this market with a known payoff vector at time t but an unknown price at time zero.

Our goal in this white paper is to determine the no-arbitrage price of our focus asset at time zero. To demonstrate the mathematics we will use the following hypothetical problem...

Our Hypothetical Problem

Assume that we have a market with three basis assets and three states of the world. We will define S_1 to be a risk-free bond and S_2 and S_3 to be risky assets. The basis assets payoff matrix at time t given the state of the world at that time is...

SOTW	S_1	S_2	S_3
1	105	80	50
2	105	120	100
3	105	160	200

The known prices of securities S_1, S_2 and S_3 at time zero are \$100, \$95 and \$92, respectively. The payoff vector at time t applicable to the focus asset is...

SOTW	Payoff
1	40
2	20
3	50

Question: What is the no-arbitrage price of our focus asset at time zero?

Method One - Redundant Asset Price Solution in Two Steps

We will define matrix $\mathbf{A}$ to be a matrix of basis asset payoffs at time t where the rows represent states of the world and columns represent basis asset payoffs given that state of the world. We will assume that the matrix columns are linearly independent such that our payoff vectors span $\mathbb{R}^m$. The equation for the basis asset payoff matrix is...

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1}$$

We will define vector $\vec{\mathbf{b}}$ to be a vector of focus asset payoffs at time t given the state of the world at that time. The equation for the focus asset payoff vector is...

$$\vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{2}$$

We will define vector $\vec{\mathbf{v}}$ to a vector of asset prices at time zero. The equation of the basis asset price vector is...

$$\vec{\mathbf{v}} = \begin{bmatrix} \text{Price of basis asset one} \\ \text{Price of basis asset two} \\ \text{Price of basis asset three} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{3}$$

We will define vector $\vec{\mathbf{w}}$ to be a vector of basis asset weights (i.e. quantities of basis assets that we long or short) at time zero. The equation of the basis asset weight vector is...

$$\vec{\mathbf{w}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \tag{4}$$

Our goal is to determine the price of our focus asset at time zero. The price of this asset will be the price that prohibits arbitrage. Since the payoffs on our three basis assets span $\mathbb{R}^3$ we can perfectly replicate our focus asset payoffs at time t by taking long and/or short positions in our basis assets at time zero. The cost to build the replicating portfolio at time zero is the price of our focus asset at time zero. The equation for our replicating portfolio in matrix:vector notation is...

$$\mathbf{A}\vec{\mathbf{w}} = \vec{\mathbf{b}} \tag{5}$$

The price of the focus asset at time zero can be determined in two steps. **Step One** is to solve for the asset weight vector. Using Equation (5) above above the solution to the asset weight vector $\vec{\mathbf{w}}$ is...

$$\textbf{Step One}: \quad \vec{\mathbf{w}} = \mathbf{A}^{-1}\vec{\mathbf{b}} \tag{6}$$

We will define the variable b_0 to be the focus asset no-arbitrage price at time zero. **Step Two** is to solve for focus asset price. Using the solution to the asset weight vector in (6) above and using the price vector $\vec{\mathbf{v}}$ per Equation (3) above the solution to focus asset price at time zero is...

$$\textbf{Step Two}: \quad b_0 = \vec{\mathbf{v}}\,\vec{\mathbf{w}}^T = \sum_{i=1}^{3} \vec{\mathbf{v}}_i \vec{\mathbf{w}}_i \tag{7}$$

Method Two - Redundant Asset Price Solution in One Step

We can rewrite the payoff matrix as a matrix of payoff vectors. Using Equation (1) above the payoff matrix can be rewritten as follows...

$$\mathbf{A} = \begin{bmatrix} \vec{\mathbf{a}}_1 \, \vec{\mathbf{a}}_2 \, \vec{\mathbf{a}}_3 \end{bmatrix} \text{ ...where... } \vec{\mathbf{a}}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \text{ ...and... } \vec{\mathbf{a}}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \text{ ...where... } \vec{\mathbf{a}}_3 = \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \tag{8}$$

Using Equation (8) above we can rewrite Equation (5) above as...

$$\vec{\mathbf{b}} = w_1\vec{\mathbf{a}}_1 + w_2\vec{\mathbf{a}}_2 + w_3\vec{\mathbf{a}}_3 = w_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + w_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + w_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \tag{9}$$

Using the basis asset prices at time zero per Equation (3) above we can rewrite Equation (9) above as...

$$\vec{\mathbf{b}} = v_1 w_1 \begin{bmatrix} a_{11}/v_1 \\ a_{21}/v_1 \\ a_{31}/v_1 \end{bmatrix} + v_2 w_2 \begin{bmatrix} a_{12}/v_2 \\ a_{22}/v_2 \\ a_{32}/v_2 \end{bmatrix} + v_3 w_3 \begin{bmatrix} a_{13}/v_3 \\ a_{23}/v_3 \\ a_{33}/v_3 \end{bmatrix} \tag{10}$$

If we make the following definitions...

$$\mathbf{A}^* = \begin{bmatrix} a_{11}/v_1 & a_{12}/v_1 & a_{13}/v_1 \\ a_{21}/v_2 & a_{22}/v_2 & a_{23}/v_2 \\ a_{31}/v_3 & a_{32}/v_3 & a_{33}/v_3 \end{bmatrix} \text{ ...and... } \vec{\mathbf{w}}^* = \begin{bmatrix} v_1 w_1 \\ v_2 w_2 \\ v_3 w_3 \end{bmatrix} \tag{11}$$

Using Equation (11) above we can rewrite Equation (5) above as...

$$\mathbf{A}^*\vec{\mathbf{w}}^* = \vec{\mathbf{b}} \text{ ...such that... } \vec{\mathbf{w}}^* = (\mathbf{A}^*)^{-1}\vec{\mathbf{b}} \tag{12}$$

Note that vector $\vec{\mathbf{w}}^*$ in Equation (12) above is now a vector of dollar investments in our basis assets rather than asset weights as in Equation (4) above. Using Equation (12) above the solution to focus asset price at time zero is...

$$b_0 = \sum_{i=1}^{3} \vec{\mathbf{w}}_i^* \tag{13}$$

Redundant Asset Price Proof

The price of the focus asset (b_0) at time t is the price that prohibits arbitrage. Using the definition of no-arbitrage above we set the conditions that the price of the focus asset must adhere to...

$$\begin{aligned} w_1 v_1 + w_2 v_2 + w_3 v_3 - b_0 &= 0 \\ w_1 a_{11} + w_2 a_{12} + w_3 a_{13} - b_1 &= 0 \\ w_1 a_{21} + w_2 a_{22} + w_3 a_{23} - b_2 &= 0 \\ w_1 a_{31} + w_2 a_{32} + w_3 a_{33} - b_3 &= 0 \end{aligned} \tag{14}$$

The Solution to Our Hypothetical Problem

Using Equation (1) above and the data from our hypothetical problem the equation for the basis assets payoff matrix $\mathbf{A}$ is...

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 105 & 105 & 105 \\ 80 & 120 & 160 \\ 50 & 100 & 200 \end{bmatrix} \tag{15}$$

Using Equation (11) above and the data from our hypothetical problem the equation for the basis assets total return matrix $\mathbf{A}^*$ is...

$$\mathbf{A}^* = \begin{bmatrix} a_{11}/v_1 & a_{12}/v_1 & a_{13}/v_1 \\ a_{21}/v_2 & a_{22}/v_2 & a_{23}/v_2 \\ a_{31}/v_3 & a_{32}/v_3 & a_{33}/v_3 \end{bmatrix} = \begin{bmatrix} 105/100 & 105/100 & 105/100 \\ 80/95 & 120/95 & 160/95 \\ 50/92 & 100/92 & 200/92 \end{bmatrix} \tag{16}$$

Using Equation (2) above and the data from our hypothetical problem the equation for the focus asset payoff vector $\vec{\mathbf{b}}$ is...

$$\vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 20 \\ 50 \end{bmatrix} \tag{17}$$

Using Equation (3) above and the data from our hypothetical problem the equation for the basis asset price vector $\vec{\mathbf{v}}$ is...

$$\vec{\mathbf{v}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 95 \\ 92 \end{bmatrix} \tag{18}$$

Method One - The solution to the price of the focus asset at time zero is...

$$\vec{\mathbf{w}} = \mathbf{A}^{-1}\vec{\mathbf{b}} = \begin{bmatrix} 1.238 \\ -1.750 \\ 1.000 \end{bmatrix} \text{ ...such that... } b_0 = \vec{\mathbf{v}}\,\vec{\mathbf{w}}^T = 49.56 \tag{19}$$

Method Two - The solution to the price of the focus asset at time zero is...

$$\vec{\mathbf{w}}^* = (\mathbf{A}^*)^{-1}\vec{\mathbf{b}} \text{ ...such that... } b_0 = \sum_{i=1}^{3} \vec{\mathbf{w}}_i^* = 49.56 \tag{20}$$

Using Equations (14) and (19) above the proof that the focus asset price at time zero is the no-arbitrage price is...

$$
\begin{aligned}
(1.238)(100) - (1.750)(95) + (1.000)(92) - 49.56 &= 0 \\
(1.238)(105) - (1.750)(80) + (1.000)(50) - 40 &= 0 \\
(1.238)(105) - (1.750)(120) + (1.000)(100) - 20 &= 0 \\
(1.238)(105) - (1.750)(160) + (1.000)(200) - 50 &= 0
\end{aligned} \tag{21}
$$

Summary: Per Equations (19), (20) and (21) above the payoffs on the focus asset at time t given the state of the world at that time can be perfectly replicated by the creation of a hedge portfolio consisting of a long position of 1.238 of asset S_1, a short position of 1.750 of asset S_2, and a long position of 1.000 of asset S_3. The cost to set up this hedge portfolio at time zero is \$49.56, which is therefore the price of the focus asset at time zero.